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Research Article

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Pattern recognition among primary school students: The relationship with mathematical problem-solving

Ioannis Rizos [1]*, Nikolaos Gkrekas [1]

[1] Department of Mathematics, University of Thessaly, Lamia, GREECE
***Corresponding Author:** ioarizos@uth.gr

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ABSTRACT

This paper explores the relationship between primary school students' pattern recognition and mathematical problem-solving. A mixed-method research approach combining worksheets and interviews is used to identify patterns in students' responses, with a focus on spatial, repeating, and growing patterns. The results of preliminary experiments with four Greek students aged eight to 11 years old, suggest a preference for geometric concepts and real-world examples. The findings could contribute to the discussions on customized pedagogical strategies in mathematics education, highlighting the importance of individualized approaches for optimal learning outcomes. The study advocates for the inclusion of visual and applied elements to cultivate critical thinking and problem-solving skills in early education.

Keywords: pattern recognition, pedagogical strategies, curricular design, problem-solving, critical thinking

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INTRODUCTION

Recognizing and utilizing patterns and structures is crucial in learning mathematics. Recent research indicates that children's abilities in pattern and structure are closely linked to their overall mathematical development. Additionally, it is possible to enhance pattern and structure competencies in young children (see Lüken & Kampmann, 2018). Patterns are expected normality, found in nature and natural sciences, such as physics and mathematics. They can also emerge from human creativity and a person's ability to process their environment, particularly in terms of sequential relationships (Hong, 2013). Pattern recognition is a concept in mathematics education that can be studied through various types of learning. It is often a part of lifelong learning, which aids in improving cognitive ability through constant observation and investigation of the system one is in (Kellman et al., 2010). Understanding the relationship between pattern recognition and mathematical abilities is important for both students learning mathematics and mathematics teachers designing their teaching pedagogies (Ling & Loh, 2023).

Critical thinking is the process of achieving a specific goal through self-construction. This concept has been incorporated into school curricula worldwide (Firdaus et al., 2015). Mathematics educators should aim to develop critical thinking skills in their students, encouraging them to use information to draw conclusions and results, rather than simply receiving it (Peter, 2012). The benefits of critical thinking are permanent. They support students in regulating their study skills and subsequently empower individuals to contribute creatively to their chosen profession (Aizikovitsh-Udi & Cheng, 2015). Additionally, researchers believe that improving critical thinking skills in mathematics students of all ages can lead to the development or major improvement in algorithmic thinking (Jablonka, 2020).

On the other hand problem-solving has emerged as an important area of mathematical research and teaching over the last four decades. It is also recognized as a crucial educational skill for elementary school students (Cornoldi et al., 2015). Research studies have heavily focused on the function of heuristics and their influence on students' problem-solving ability (Mousoulides & Sriraman, 2020). The term "problem-solving" refers to engaging in a task for which the solution method is not known in advance (National Council of Teachers of Mathematics [NCTM], 2000).

The aim of this paper is to shed some light on the complex interplay between primary school students' intrinsic pattern-seeking and its impact on their problem-solving abilities. Our goal is to contribute to the improvement of related pedagogical practices, since such a perspective is of interest to the Greek educational system, but not much work seems to have been done in this area.

The main research question is whether and how the engagement of primary school students with pattern recognition can aid in the development of mathematical problem-solving skills. From formulating a worksheet with different types of patterns, such as spatial and alpharithmetic, and a follow-up interviews, we might actually be able to interpret some students' responses, categorize their ways of thought and then potentially reshape the educational process to avoid possible misconceptions and improve their problem-solving abilities.

To enable a more rigorous reformulation of the above question in a future study with a larger sample, we conducted an initial empirical study with four elementary school students who voluntarily participated. Employing a mixed research methodology, we collected quantitative and qualitative data in the form of worksheets and interviews, respectively. Although further research is needed, the initial findings suggest that a teaching approach including hands-on activities is beneficial. This highlights the importance of manipulatives in enhancing understanding of mathematical concepts and aiding problem-solving. It is of great importance to observe, understand, or even build such relationship in a more systematic manner, as the potential for future educational methods to evolve from such research and its extensions in considerable. The incorporation of problem-solving and patterns into the curriculum could pave the way for transformative era in the field of primary mathematics education.

THEORETICAL FRAMEWORK

Critical thinking is a form of thinking in which a person questions, analyses, evaluates and judges what comes to his or her attention. As a skill it needs to be developed, practiced and continuously integrated into the curriculum to engage students in active learning (Radulović & Stančić, 2017). Research done in Norwegian schools in early grades has shown that the ability to have mathematical perspective of problems can positively affect the critical thinking ability of the students (see Sachdeva & Eggen, 2021). To support this assumption, focused attention must be paid to the implementation of content, the learning process and assessment methods. In terms of content implementation, particularly in mathematics, teaching techniques that promote memorization do not support critical thinking. Although some content, such as vocabulary definitions, require memory, it is the application of content that stimulates thinking (Snyder & Snyder, 2008). Similar processes can be applied for prospective teachers also (Kozikoglu, 2019). The aforementioned researchers contributed to the understanding of the relationship between critical thinking in mathematics education and teaching practice, which is utilized in our paper, by conducting the worksheets and questions in their original form. One of the key variables in the development of problem-solving skills is critical thinking, which encompasses the concept of rigorous mathematical reasoning. Indeed, mathematical reasoning constitutes a subset of critical thinking.

Mathematical reasoning is the process of deriving conclusions from logical mathematical premises based on relevant facts and sources that are assumed to be true. The discourse on mathematical reasoning within the mathematics education research community is not monolithic and is not dominated by a single voice. Various perspectives on mathematics, as well as teaching and learning, are presented (Jeannotte & Kieran, 2017). Reasoning is a mathematical ability with complex implications, including abilities that are not easily attainable by students. The quality of students' mathematical reasoning is still dominated by imitative reasoning, where the faced problematic situations of the student are fixated on the application of routine in daily lessons (Sukirwan et al., 2018). Mathematical reasoning is fundamental for mathematical problem-solving and even for pure mathematics, such as formal proofs and analytic methods (Lithner, 2000). Students are able to do reasoning if they are able to use reasoning skills in patterns and traits, manipulate mathematics in generalizing or explain mathematical ideas and statements (Hasanah et al., 2019). Results of studies showed that learning through problem-solving strategy was more effective than the scientific approach to students' mathematical abilities in communication, creativity, problem-solving, and mathematical reasoning (Tambunan, 2019). These results were crucial to form certain expectations when we conceived the main idea of our research, enriching the worksheet with realistic problems considering mathematical entities, as figures, numbers, etc.

Other studies have shown that the effect of serial multiple mediation on predicting problem-solving ability from self-directed learning, academic self-efficacy, and self-regulated learning was significant, meaning that the ability of predicting strategies is directly linked to problem-solving abilities and pattern recognition (Hwang & Oh, 2021; Monteiro et al., 2020). Our research methods are similar to this, meaning that we left the participants to properly answer the worksheet and the questions as pleased, although in strictly monitored, but not an interfered environment. Research has been conducted at the middle-school level and a correlation analysis was used to determine the relationship of variables on non-routine problem-solving skills, and the predictive effects of the predictor variables on non-routine mathematics problem-solving skills were examined (Ozturk et al., 2020). Those skills do not have a specific field to concentrate on; problem-solving and identifying patterns can be applied with many different ways, alpharithmetic, spatial and more.

Spatial patterns, which refer to the arrangement of objects or events in space, are often used as (standard) number presentations to visualize numerical structures in a specific geometrical way (Lüken & Kampmann, 2018). These patterns can range in levels of complexity. Commonly, a repeating pattern that has an AB discernible unit of repeat is introduced to students (e.g., ABABABABA; jump, clap, jump, clap; night, day, night, day) (Miller, 2019). In our worksheet we include such a case (see **Appendix A**). Previous research has taken a significant part in our inspiration in constructing the worksheet and coordinating our research properly.

As resent research has shown, problem-solving in open-ended problems (problems that have several path to resolution or more than one valid answer) can improve the intuitional analysis of the participants and get the engagement to the task (Rizos & Gkrekas, 2023a). Multiple approaches on solving a problem or proving a conjecture can enrich the problem-solving abilities of a person and introduce him to other methods, like the programming languages of the dynamical geometry environments in order to test a theory, a conjecture or a mathematical problem, in general (Rizos & Gkrekas, 2023b). These utilities were not integrated in our research, other than the tool of making the graphics for the worksheet; but it shares the premise of making a conjecture based on the participants' intuition and observation.

Brief notes on Greek Educational Reality

In the new curriculum textbooks in the Greek educational system, the terms of *problem-solving* and *pattern seeking* are introduced, even in primary education. The field of elementary statistics and probabilities is also introduced to young Greek students in the concept of normality in which topic there has also been conducted research on pre-school activities (see Skoumpourdi et al., 2009). In the curricula patterns start from the usual chapter of symmetry and then they continue with the premise of series and sequences (such as triangular numbers) and geometric representations.

In the form of shapes and sizes different formations are given as activities to students throughout the school year. In later years, in high school, arithmetic patterns are the part of curriculum, concentrating on the "algebra and number theory" aspect, in contrast to the patterns found in primary school, concentrated on geometry.

Following a prolonged period of formalistic mathematics teaching in Greek secondary education, the new curricula now emphasize algebraic operations, solution techniques and identifying the "correct" pattern. Perhaps this emphasis is answerable for many students' uncritical responses to questions testing their concept images. For example in a six-month research project involving sixty prospective mathematics teachers, a common description of rational numbers was that, if the decimal representation of a number does not end, then, in order that this number is rational it is necessary and sufficient that its decimal representation *follows a pattern*, otherwise it is irrational (Rizos & Adam, 2022). Some children seem to have completely related the term of patterns with simple direct repetition or mirroring, which is already a significant chapter in symmetry and pattern recognition in mathematics, but is not unique (Mason, 2014). This might have been a result of traditional education in early grades. Such patterns can be interrelated with the concept of generalization, which may not be able to form properly in a traditional classroom, in contrast to active learning (Cohn et al., 1994).

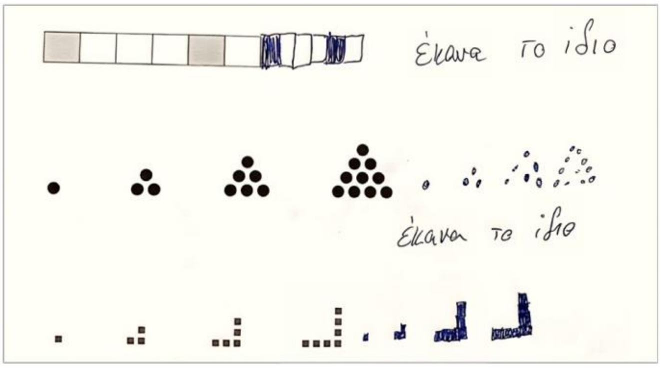

Figure 1. How Jim continued geometric patterns on worksheet (Source: Authors)

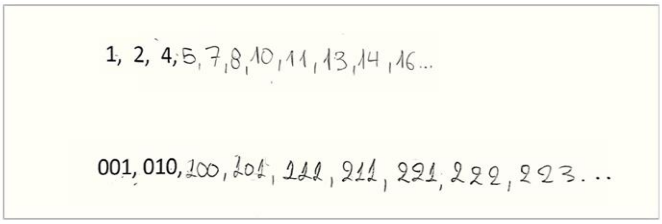

Figure 2. How Maria continued sequences on worksheet (Source: Authors)

METHODOLOGY

Studying certain topics in mathematics education poses a challenge. Although written tasks can yield valuable data on students' results, they do not shed light on the thought processes that led to these results. In our research, we employed a mixed research methodology that combines detailed qualitative data with generalized quantitative data (Creswell & Plano Clark, 2011; Tashakkori & Teddlie, 2010).

Specifically, we followed the convergent parallel design, which involves applying quantitative (worksheets) and qualitative (interviews) data simultaneously during the research process. The methods were given equal priority, and the data was analyzed independently. The two sets of results were then merged, and an overall interpretation was correlated (Creswell & Plano Clark, 2011).

The principle of informed consent of the subjects (Mertler, 2012) was adhered to for the ethics of the research, as consent was secured from the students and their parents, and a commitment was made at the time of writing that the principles of privacy, anonymity and confidentiality would be respected (Cohen et al., 2018).

FINDINGS

As an initial step towards developing mathematical problem-solving skills through engaging students in pattern recognition, we conducted a pilot worksheet (see **Appendix A**) in late 2023 with four primary school students in Greece (two boys and two girls, aged eight-11 years old), who volunteered to participate outside of formal lessons. The data was collected in a fully monitored environment, ensuring the participants did not experience any anxiety thinking this was a formal test and any influence from external sources, other than the instructor for coordination purposes and interviews. We are positive that in case the experiment was repeated multiple times the results would be similar, because of the specific conditions it was held in.

Below, we briefly present their responses:

Jim, a 5th grade student, completed the sequence 1, 2, 4, …, as follows: 1, 2, 4, 6, 8, 10. When we asked him how his answer came out, he said that he "added two".

In the patterns that had a figure, Jim simply repeated the figures (see **Figure 1**). In fact, before he wrote anything, he asked us:

> Is this a pattern?
>
> Let's say, yes.
>
> Ok then! I will repeat the same.

A 6th grade student named Maria used her own "additive rule" to continue the first sequence. When asked, she explained that the rule involves adding one in the first step and two in the second step, which is then repeated. For the next sequence, she stated that the digit 1 is shifted to the left and then 1 is added to the existing digits, always from right to left (see **Figure 2**).

Joseph, a 4th grade student, successfully completed the four geometric patterns on the worksheet but struggled with the three questions involving alphanumeric sequences. When asked about his difficulty, he stated that,

> "In the first row I understand what I have to do. I add two at a time, so I will then write 6, 8, 10, etc. For the following rows I do not know".

The subsequent conversation went, as follows:

> If the figure you see extends indefinitely in the same way, what color would the 13th square be? Would it be white or grey?

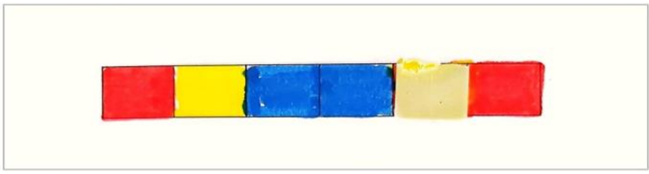

Figure 3. Boxes, as they were completed by Alexia (Source: Authors)

Uh … it would be grey.

How did you find it?

There's a pattern here, which repeats every four squares. Of these, the first one is grey and the other three are white. So, 4+4+4=12 and 12+1=13. So it is grey.

Can you find out what color the 82^{nd} square will be?

Uh… Well, it's 20·4=80 and 80+2=82. The second square is white, so the 82^{nd} square will also be white.

So what color will the 937^{th} square be?

Oh, that's a big number, but if I do multiplication by four I'll get it.

Finally, a 3^{rd} grade student named Alexia did something unexpected, but really unique. She altered the question itself, specifically the one with the boxes (as seen in **Figure 3**), in order to create a pattern of her choice, arbitrarily. She erased with correction fluid the shading of the fifth square and she also added color to the boxes, thus creating another pattern of her choice.

Table 1 is a summary of the strategies developed by the students to continue the seven patterns of the worksheet. In **Table 1**, we categorize the strategies of the young participants, and we notate them, as following (with the notation being (S)trategy No. of pattern(.) No. of strategy):

– S1.1=1, 2, 4, 6, 8, …
– S1.2=1, 2, 4, 5, 7, 8, …
– S2.1=001, 010, 0100
– S2.2=001, 010, 020, 030, ...
– S2.3=001, 010, 101, 111, …
– S3.1=AA, BB, CC, …
– S4.1= …, yellow circle, blue triangle
– S4.2=Simple full repetition of the given pattern
– S5.1=…, white box, white box, grey box
– S5.2=Simple full repetition of the given pattern
– S5.3=Created their own pattern with colors and mirroring
– S6.1=Next triangular number (as more expected)
– S6.2=Simple full repetition of the given pattern
– S7.1=Next shape (as more expected)
– S7.2=Simple full repetition of the given pattern

After briefly conversing with each participant individually, we encouraged them to approach the last pattern from a different perspective (refer to **Appendix A**). Specifically, we restructured the dots by rotating them to form a sequence of squares (refer to **Appendix B**). We then asked the students to identify the type of pattern created (expecting them to answer "square") and guess the pattern's type and size in the next step. In this context, all students provided the same response, that continuing in the same manner would always result in squares. When asked how they arrived at this conclusion, they unanimously agreed on the square dimensions of the shapes (2×2, 3×3, etc.). The children were able to comprehend that the squares were made up of 1, 4, 9, and 16 dots, respectively and that the next square would contain 25 dots (5×5). This observation could potentially lead to the development of a *formal model*, specifically that the sum of n odd numbers equals n^2. Each child was individually assessed, completing a questionnaire and participating in a discussion, which took approximately 45 minutes in total.

Table 1. Students' strategies

Participants	P 1	P 2	P 3	P 4	P 5	P 6	P 7
Jim	S1.1	S2.2	S3.1	S4.2	S5.2	S6.2	S7.2
Maria	S1.1	S2.3	S3.1	S4.1	S5.1	S6.1	S7.1
Joseph	S1.1	-	-	S4.1	S5.1	S6.1	S7.1
Alexia	S1.2	S2.1	S3.1	S4.1	S5.3	S6.1	S7.1

Note. P: Pattern

It is suggested that the intervention may be more effective and its results more noticeable if manipulative materials are used. Also, restricted sample size used in this study prevents us from generalizing about wider population. Nevertheless, results show a clear relationship or at least the potential for one, between pattern recognition and the development of problem-solving strategies in elementary school students. The responses give an insight towards the answer to the main research question of the research, signifying a distinct relation regarding different responses on different types of patterns. In addition, it seems that students have a very well rooted personal idea of what a pattern is (either it is scientifically sound or not) and they actively try to form these types of patterns that they think about. Also, students showed a lack of development in their mathematical reasoning abilities. The experiment comprised a series of simple tasks with more than one valid responses with minimal necessary guidance by one observer, a way of simplified open-ended problem-solving.

DISCUSSION

Using a mixed-method research approach, we examine curricular materials as well as students' answers to various difficulties to uncover hidden patterns. Real-world teaching examples are combined with targeted problem-solving activities to relate students' pattern detection and problem-solving abilities and assist them create an objective perspective. In the work with four primary school children, three main types of mathematical patterns are used: spatial structure patterns, repeating patterns, and growing patterns.

In conducting a preliminary experiment with four young participants (aged eight to 11), the findings suggest a notable inclination among children towards geometric concepts and natural, applied examples, as opposed to numerical and algebraic representations. This observation aligns with existing research in mathematics education, emphasizing the intuitive appeal of visual and tangible elements in the early stages of learning (Cybulski et al., 2015). The participants demonstrated a heightened familiarity and engagement with geometric ideas, possibly due to the inherent visual nature of shapes and spatial relationships. Children may rely on spatial skills to complete repeating patterning tasks, especially when the tasks include working with visual patterns constructed with objects (Rittle-Johnson et al., 2019).

Natural, applied examples further captured their attention, highlighting the significance of real-world contexts in fostering mathematical understanding. This research aligns with our work, which has employed visual patterns, such as shapes and colors, as well as everyday examples of routines, to examine how these factors influence the way participants' pattern-seeking brains operate. However, to the best of our knowledge, no comprehensive research on this topic has been conducted to enable a direct comparison of our findings with those of others. In contrast, the somewhat reduced enthusiasm for numerical and algebraic aspects highlights potential challenges or lack of familiarity that young learners may face when introduced to abstract symbols and numerical expressions, as confirmed in **Table 1**.

Another phenomenon observed was the different approach to the "expected" geometric sequence (i.e., 1, 2, 4, …) when three out of four participants recognized it as the sequence of even numbers, disregarding the *first term*–which seems to inductively relate with the way the Pythagoreans considered such problems, for whom the *unit* (*μονάς*) was something sacred (see Van Der Waerden, 1975). These insights support an instructional approach that strategically incorporates visual and applied elements to enhance the accessibility and appeal of mathematical concepts for children at an early age. The observed patterns provide a basis for further investigation and curriculum development. It is important for mathematics education to consider the cognitive preferences of young learners.

Research has shown that with the use of the appropriate manipulative material in real-world problems can help student escape misconceptions (like the illusion of linearity) and understand mathematical concepts better (see Rizos & Foykas, 2023). This way, when teaching students to recognize patterns, hands-on activities can make the mathematical terms and sequences easier to be grasped. Using real-world problems can also positively affect the problem-solving abilities of children, because of the natural perspective those examples give (the thought process can be easily verified by a mere observation and be imprinted that way in the mind of the observer). All those statements are a small part of problem-solving and problem-posing strategies, and in the problem-solving experience changing strategies midway is encouraged, as oxymoronic as it may seem with classic mathematical problem-solving methods (see Polya, 1945).

The traditional teaching model, as is the one that undermines a specific relation between instructor and learner as the one that dictates and the one that passively accepts the dictated knowledge, has various limitations. It concentrates on a strict and possibly incomplete definition of patterns and the curriculum is restrained in some specific, oriented inside-the-box examples, which is leading to multiple fallacies, misconceptions in the children's perception on pattern recognition. Most of the restricted examples that the traditional model seems to use have marked as "acceptable" result the idea of "1-1" simple repetition of the sequences and series of shapes or basic mirror symmetry (as shown by the data collected at our research) and this has probably worsen or even impair the young students' ability to observe, comprehend critically and formulate everyday patterns (or even more complex and advanced), different that the ones instructed in the classroom. The traditional teaching method does not encourage the use of open-ended problems. Multiple valid answers is (falsely) said to complicate the class and mess with teacher's authority, causing confusion. This conventional way of thought is embedded in everyday school practice, in contrast to our suggestions and to the data verifying the existence of multiple valid answers to one sequence/ pattern formation.

As an idea, the change of strategies seems counterintuitive, but as seen in practice, it is the intuition that guides the participant to change their strategy. In application, some of the suggestions in order to improve modern primary mathematics education by utilizing the results of the present research are focused on more hands-on-integration of tools. A specific suggestion is to utilize in kindergarten ICT, shapes and prisms in real life and in computer form in order to teach young students the concept and the method of recognizing multiple types of patterns and structures appearing in the surrounding environment. This approach has the potential to enhance young children's pattern skills, a conclusion that is supported by previous research (Lüken & Kampmann, 2018).

Another suggestion is the use of important everyday observation of cycles (like day and night, traffic lights, buildings etc.) in order to allow students to study natural occurring normalities and not let them believe that the existence of pattern stops on a strict formal definition or the use of simple repetition (or even mirroring symmetry). Such useful examples are used in our research worksheet and it extension (see **Appendix A** and **Appendix B**).

To answer our research question, we would say that the research data gathered from interviews and worksheets indicates that everyday problems and patterns can have a positive impact on children's problem-solving skills and cognitive abilities, which is important for both students and teachers (cf. Ling & Loh, 2023). However, conventional teaching methods can lead to a misunderstanding of the concept of the pattern. This pilot study has collected useful information that could lead to a larger, quantitative questionnaire with more categories of questions and responses to different types of patterns. The hypothesis is that the examples and teaching techniques in the pattern lesson can improve the critical thinking skills and mathematical reasoning of young learners.

CONCLUSIONS

This investigation highlights the importance of patterns in developing critical thinking abilities and achieving optimal results in mathematics education. It emphasizes the relationship between pattern recognition in primary school children and mathematical problem-solving. The incorporation of spatial, recurring, and increasing patterns into the curriculum is a critical method for encouraging active learning. Our preliminary studies with primary school students indicate a strong preference for geometric concepts and real-world applications. This underlines the necessity of incorporating visual and practical components into early schooling, which remains to be explored by future research. Future research may include the incorporation of such worksheets into the traditional teaching process; to extend the experiment in a more diverse group of students, making quantitative analysis on the results, study in different age groups, different types of schools etc. The trends discovered among pupils underscore the importance of adapting mathematics instruction to match cognitive preferences and developmental stages. In addition, the findings endorse a teaching approach that includes hands-on activities, emphasizing the significance of manipulative materials to enhance understanding of mathematical concepts and aid problem-solving.

The research contributes to the ongoing discussion of customized pedagogical practices aimed at improving the quality of mathematics education and stimulating critical thinking skills from the early stages of learning. This research contributes to the ongoing debate on the most effective teaching methods for improving mathematical intuition and problem-solving skills in primary students. The incorporation of such tasks into the kindergarten and primary school curriculum may result from the findings of this research and serve to stimulate debate among educators about the most effective pedagogical approaches to modern mathematics education.

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Declaration of interest: The authors declare no competing interest.

Data availability: Data generated or analyzed during this study are available from the authors on request.

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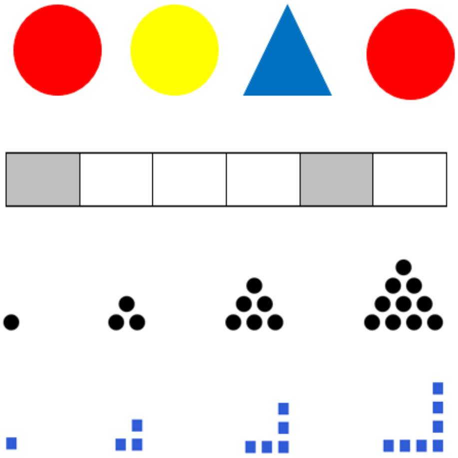
APPENDIX A

Continue the following rows with the numbers, letters, figures, or colors that you think match better:

1, 2, 4, …

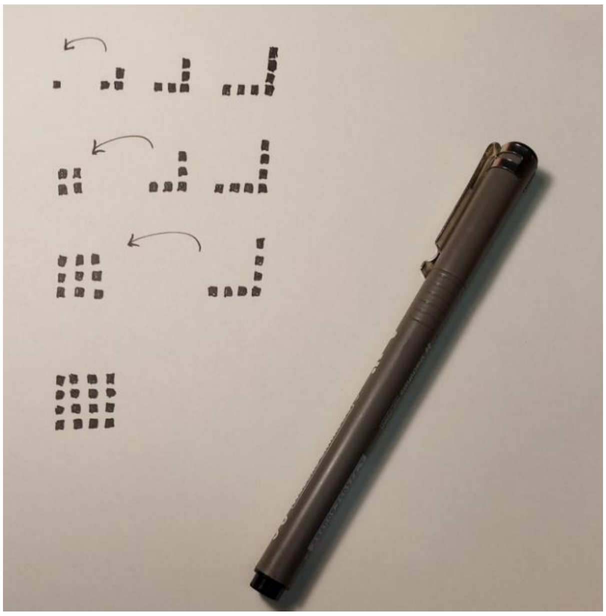
001, 010, …

A, A, B, …



(Source: Authors)

APPENDIX B



(Source: Authors)